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Nonlinear Receding-Horizon State Estimation by Real-Time Optimization Technique

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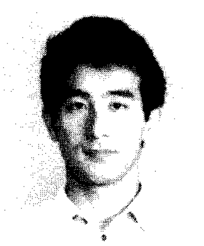
A real-time optimization technique is proposed for optimal state estimation of general nonlinear systems. A state estimation algorithm is determined so that a receding-horizon performance index is minimized. Application of the stabilized continuation method results in a real-time solution technique that does not involve any approximation or iterative algorithms in principle. The discretization of the algorithm is the only approximation required in the actual implementation on digital computers. The structure of the state estimation algorithm is clarified based on the present solution technique, and a simple model of a nonlinear space vehicle is employed as an application example. Results of simulation and experiment validate the effectiveness of the proposed algorithm.

Introduction

NONLINEAR optimal feedback control has various applications in engineering, including aerospace engineering. Once a performance index and a mathematical model are given, optimization yields a feedback law that achieves the best possible performance in terms of the given performance index. Most control design problems can be automated except for choosing the performance index and the mathematical model, if the nonlinear optimal feedback control theory is implemented. A well-known difficulty of the nonlinear optimal feedback control is that it results in nonlinear two-point boundary-value problems (TPBVPs) that, in general, cannot be solved analytically. Therefore, exploration of solution algorithms is a key problem to establish a practical technique of nonlinear optimal feedback control.

Recently, the authors[1,2] proposed a real-time optimization technique for the nonlinear receding-horizon state-feedback control, which is a kind of nonlinear optimal feedback control. Performance indices in the receding horizon control have a moving initial time and a moving terminal time. Since the time interval in the performance index is finite, the optimal feedback law can be determined even for a system that cannot be stabilized in the sense of Lyapunov with any feedback laws. Therefore, the receding-horizon control problem can deal with a broader class of optimal feedback control problems than infinite-horizon regulator problems. The nonlinear receding-horizon control problem requires solutions of a family of nonlinear TPBVPs. The solution technique proposed in Refs. 1 and 2 is based on the stabilized continuation method for solving optimal control problems,[3] and does not involve any iterative algorithms. It is shown that the algorithm can be implemented in a hardware experiment.

This paper discusses real-time algorithms of the nonlinear receding-horizon state-estimation problem as a counterpart of the nonlinear receding-horizon state-feedback control problem. Nonlinear state estimation is as important as nonlinear state-feedback control to establish a general theory of nonlinear feedback control, since it is often the case in practice that states are only partially known as sensor data. A nonlinear state estimation technique is also available for parameter estimation of nonlinear systems by regarding unknown parameters as unknown constant states. Therefore, not only output feedback control but also adaptive control of nonlinear systems is possible by combining nonlinear state estimation



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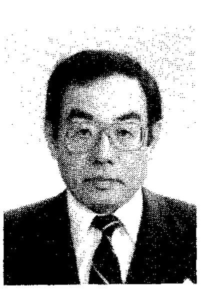

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with nonlinear state feedback control, although there is no performance or stability guarantee. This paper formulates the nonlinear receding-horizon state-estimation problem as the dual of the nonlinear receding-horizon state-feedback control problem. Linear receding-horizon state estimation has been studied by Thomas[4] and has been shown to result in an algorithm that is similar to the Kalman filter. For optimization-based nonlinear state estimation problems, only iterative algorithms are available,[5–7] and efficient algorithms have not been established for practical fast nonlinear dynamic systems. This paper derives noniterative real-time algorithms for the formulated problem through the same approach as the receding-horizon state-feedback. The proposed algorithm is applied to a simplified model of a space vehicle and is examined in numerical simulation. The algorithm is also implemented in a hardware experiment, and is shown to be suitable for real-time nonlinear state estimation.

Problem Formulation

The dynamical system treated here is expressed in the following differential equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{1}$$

where $\boldsymbol{x}$ denotes the state and $\boldsymbol{u}$ the input to the system. The input $\boldsymbol{u}(t)$ is assumed to be known. A time-variant system can also be expressed in this form by regarding the time as a state of the system. In this paper, motivated by the receding-horizon state-feedback control, an optimal state estimation algorithm is determined so as to minimize a receding-horizon performance index:

$$J = \eta[\boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{u}(t)] + \varphi[\boldsymbol{x}(t-T), \boldsymbol{y}(t-T), \boldsymbol{u}(t-T)] + \int_{t-T}^{t} L[\boldsymbol{x}(\tau), \boldsymbol{y}(\tau), \boldsymbol{u}(\tau)]\,\mathrm{d}\tau \tag{2}$$

where $\boldsymbol{y}(t)$ denotes the measured output of the actual system. Functions η, φ, and L penalize the residual between measured and estimated output. Note that the performance index evaluates the residual from the finite past $t - T$ to the present time t. The present state $\boldsymbol{x}(t)$ is optimized to minimize the performance index.

The receding-horizon state-estimation problem can be converted to a family of optimal state estimation problems parameterized by time t as follows.

Minimize:

$$\eta[\boldsymbol{x}^*(0,t), \boldsymbol{y}(t), \boldsymbol{u}(t)] + \varphi[\boldsymbol{x}^*(-T,t), \boldsymbol{y}(t-T), \boldsymbol{u}(t-T)] + \int_{-T}^{0} L[\boldsymbol{x}^*(\tau,t), \boldsymbol{y}(t+\tau), \boldsymbol{u}(t+\tau)]\,\mathrm{d}\tau \tag{3}$$

Subject to:

$$\frac{\partial \boldsymbol{x}^*(\tau,t)}{\partial \tau} = \boldsymbol{f}[\boldsymbol{x}^*(\tau,t), \boldsymbol{u}(t+\tau)] \tag{4}$$

where $()^*$ denotes a variable in the converted optimal state estimation problem to distinguish it from its correspondence in the original problem. For a fixed time t, the trajectory of the state $\boldsymbol{x}^*$ is estimated along the fictitious time τ. The estimate of the present state $\boldsymbol{x}(t)$ is given by the state at $\tau = 0, \boldsymbol{x}^*(0,t)$, minimizing the performance index. The first-order necessary conditions for the optimal solution are readily obtained as a TPBVP by the calculus of variations as[8]

$$\frac{\partial \boldsymbol{x}^*(\tau,t)}{\partial \tau} = H_\lambda^T \tag{5}$$

$$\frac{\partial \boldsymbol{\lambda}^*(\tau,t)}{\partial \tau} = -H_x^T \tag{6}$$

$$\boldsymbol{\lambda}^*(0,t) = \eta_x^T[\boldsymbol{x}^*(0,t), \boldsymbol{y}(t), \boldsymbol{u}(t)] \tag{7}$$

$$\boldsymbol{\lambda}^*(-T,t) = -\varphi_x^T[\boldsymbol{x}^*(-T,t), \boldsymbol{y}(t-T), \boldsymbol{u}(t-T)] \tag{8}$$

where $\boldsymbol{\lambda}^*(\tau,t)$ denotes the costate, H denotes the Hamiltonian defined as

$$H = L + \boldsymbol{\lambda}^{*T}\boldsymbol{f} \tag{9}$$

and H_x denotes the partial derivative of H with respect to $\boldsymbol{x}^*$ and so on. Note that the costate $\boldsymbol{\lambda}^*$ is specified at $\tau = 0$ by Eq. (7), whereas the state $\boldsymbol{x}^*$ is unknown at $\tau = 0$. The roles of the state and costate are interchanged in the optimal state estimation problem compared to the optimal control problem.

Since the state and costate at $\tau = -T$ are determined by Eqs. (5) and (6) from the state and costate at $\tau = 0$, the TPBVP can be regarded as a nonlinear equation with respect to the state at $\tau = 0, \boldsymbol{x}^*(0,t)$, as

$$\boldsymbol{F}[\boldsymbol{\lambda}^*(0,t), \boldsymbol{x}^*(0,t), t, T] = \boldsymbol{\lambda}^*(-T,t) + \varphi_x^T[\boldsymbol{x}^*(-T,t), \boldsymbol{y}(t-T), \boldsymbol{u}(t-T)] = 0 \tag{10}$$

In the next section, the stabilized continuation method is introduced to solve the nonlinear equation (10) in real time.

Real-Time Solution by Stabilized Continuation Method

Stabilized Continuation Method

To solve the optimal state estimation problem in real time with moderate data storage, this paper employs a noniterative solution technique, the stabilized continuation method. The problem is converted to an initial value problem of an ordinary differential equation that can be solved numerically without iterative methods. A general discussion of the stabilized continuation method is given in Ref. 3 for solving optimal control problems with general boundary constraints and nondifferential equality constraints.

Since the nonlinear equation (10) has to be satisfied at any time t, $\mathrm{d}\boldsymbol{F}/\mathrm{d}t = 0$ holds along the trajectory of the optimal estimate. The differential equation of the optimal estimate $\boldsymbol{x}(t) = \boldsymbol{x}^*(0,t)$ is derived from that condition. The ordinary differential equation of $\boldsymbol{x}(t)$ can be integrated with numerical algorithms without recourse to any iterative approximation methods. However, numerical error in the solution may accumulate through the numerical integration in practice, and some correcting techniques are necessary in the numerical algorithm. This paper employs the stabilized continuation method[3] so that the error attenuates as the integration proceeds as follows:

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t} = -\zeta \boldsymbol{F} \tag{11}$$

where ζ denotes a positive scalar. Equation (11) results in the exponential attenuation of the error.

Algorithm Based on Backward Sweep: Method I

To evaluate the derivative of $\boldsymbol{x}(t) = \boldsymbol{x}^*(0,t)$ with respect to time, we consider the variation in the optimal solution caused by the infinitesimal variation in the time t. It is straightforward to show that the variations in the state and costate are governed by the following linear differential equation[8]:

$$\frac{\partial}{\partial \tau}\begin{bmatrix} \delta\bar{\boldsymbol{x}}(\tau,t) \\ \delta\bar{\boldsymbol{\lambda}}(\tau,t) \end{bmatrix} = \begin{bmatrix} A(\tau,t) & 0 \\ -C(\tau,t) & -A^T(\tau,t) \end{bmatrix}\begin{bmatrix} \delta\bar{\boldsymbol{x}}(\tau,t) \\ \delta\bar{\boldsymbol{\lambda}}(\tau,t) \end{bmatrix} \tag{12}$$

where

$$\delta\bar{\boldsymbol{x}} = \delta\boldsymbol{x}^* - \boldsymbol{f}\,\mathrm{d}t, \qquad \delta\bar{\boldsymbol{\lambda}} = \delta\boldsymbol{\lambda}^* + H_x^T\,\mathrm{d}t \tag{13}$$

$$A = f_x, \qquad C = H_{xx} \tag{14}$$

The variation in the function $\boldsymbol{F}$ is expressed as

$$\mathrm{d}\boldsymbol{F} = \delta\bar{\boldsymbol{\lambda}}(-T,t) + \varphi_{xx}|_{\tau=-T}\,\delta\bar{\boldsymbol{x}}(-T,t) + \left(1 - \frac{\mathrm{d}T}{\mathrm{d}t}\right)\left(\varphi_{xx}\boldsymbol{f} + \varphi_{xy}\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} + \varphi_{xu}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} - H_x^T\right)\Bigg|_{\tau=-T}\mathrm{d}t \tag{15}$$

The variations $\delta\bar{\boldsymbol{x}}$ and $\delta\bar{\boldsymbol{\lambda}}$ are expressed in terms of the transition matrix of Eq. (12), and the differential equation of $\boldsymbol{x}(t) = \boldsymbol{x}^*(0,t)$ can be derived from Eqs. (11) and (15). However, algorithms can also be obtained through the backward sweep method,[8] and the algorithms based on the backward sweep method require less computational load than the algorithm based on the transition matrix.[2]

In the backward sweep method for the receding horizon control, the variation in the costate is expressed in terms of other variations along the optimal trajectory.[2] Since the roles of the state and costate are interchanged in the optimal state estimation problem compared to the optimal control problem, the variation in the state is assumed to be expressed in terms of other variations in the receding-horizon state-estimation problem as

$$\delta\bar{x}(\tau, t) = \Lambda^*(\tau, t)\delta\bar{\lambda}(\tau, t) + r^*(\tau, t)\,\mathrm{d}t \tag{16}$$

where matrices $\Lambda^*(\tau, t)$ and $r^*(\tau, t)$ have to satisfy the following terminal condition derived from Eqs. (11) and (15):

$$\Lambda^*(-T, t) = -\varphi_{xx}^{-1}\big|_{\tau=-T} \tag{17}$$

$$r^*(-T, t) = \varphi_{xx}^{-1}\big|_{\tau=-T}\left[-\zeta F - \left(1 - \frac{\mathrm{d}T}{\mathrm{d}t}\right) \times \left(\varphi_{xx} f + \varphi_{xy}\frac{\mathrm{d}y}{\mathrm{d}t} + \varphi_{xu}\frac{\mathrm{d}u}{\mathrm{d}t} - H_x^T\right)\bigg|_{\tau=-T}\right] \tag{18}$$

Furthermore, Eq. (12) implies that matrices $\Lambda^*(\tau, t)$ and $r^*(\tau, t)$ have to be governed by the following differential equations:

$$\frac{\partial \Lambda^*(\tau, t)}{\partial \tau} = A\Lambda^* + \Lambda^* A^T + \Lambda^* C \Lambda^* \tag{19}$$

$$\frac{\partial r^*(\tau, t)}{\partial \tau} = (A + \Lambda^* C) r^* \tag{20}$$

The optimal trajectory is obtained for each time t by integrating the Euler–Lagrange equation, Eqs. (5) and (6), forward along τ, and the differential equations of Eqs. (19) and (20) are integrated backward along the optimal trajectory. Since Λ^* is symmetric, Eq. (19) reduces to an $n(n+1)/2$-dimensional differential equation for an n-dimensional system. The estimate of the state is updated by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left(I - \Lambda^*\eta_{xx}\right)^{-1} \times \left[f + \Lambda^*\left(\eta_{xy}\frac{\mathrm{d}y}{\mathrm{d}t} + \eta_{xu}\frac{\mathrm{d}u}{\mathrm{d}t} + H_x^T\right) + r^*\right]\bigg|_{\tau=0} \tag{21}$$

If the penalty η is not included in the performance index, the algorithm is simplified as follows:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f[x(t), u(t)] + \Lambda^*(0, t) L_x^T[x(t), y(t), u(t)] + r^*(0, t) \tag{22}$$

The second term on the right-hand side corresponds to feedback of the residual, and the structure of the receding-horizon state estimation is similar to the structure of the extended Kalman filter[9] (EKF) for nonlinear estimation. However, in contrast to the EKF, the present state estimation problem is deterministic without any disturbances or noise, and it minimizes the receding-horizon performance index. The third term on the right-hand side of Eq. (22) results from the moving terminal time and correction of the error in the optimality condition. Furthermore, the matrix $\Lambda^*(0, t)$, which corresponds to the approximate estimation error covariance matrix in EKF, is determined in a more complicated manner than EKF. Whereas a differential Riccati equation is integrated in parallel with the state estimation equation in EKF, the differential Riccati equation (19) has to be solved over $-T \le \tau \le 0$ at each time t in the receding-horizon state estimation.

Nonsingularity of φ_{xx} is required at $\tau = -T$ in the preceding algorithm, as is shown in Eqs. (17) and (18). This condition rarely holds in general state estimation problems. The penalty φ usually consists of the residual between measured and estimated output and does not include a state that does not appear in the output directly. The derivative of φ vanishes with respect to such a state that does not appear in the output, and φ_{xx} is singular.

Algorithm Based on Backward Sweep: Method II

Since the preceding algorithm is not available when φ_{xx} is singular, which is often the case in the state estimation problem, we consider another algorithm based on the backward sweep method. In contrast to the preceding algorithm, the variation in the costate is expressed in terms of other variations in the present algorithm,

$$\delta\bar{\lambda}(\tau, t) = S^*(\tau, t)\delta\bar{x}(\tau, t) + c^*(\tau, t)\,\mathrm{d}t \tag{23}$$

where matrices $S^*(\tau, t)$ and $c^*(\tau, t)$ have to satisfy the following terminal condition derived from Eqs. (11) and (15):

$$S^*(-T, t) = -\varphi_{xx}|_{\tau=-T} \tag{24}$$

$$c^*(-T, t) = -\zeta F - \left(1 - \frac{\mathrm{d}T}{\mathrm{d}t}\right) \times \left(\varphi_{xx} f + \varphi_{xy}\frac{\mathrm{d}y}{\mathrm{d}t} + \varphi_{xu}\frac{\mathrm{d}u}{\mathrm{d}t} - H_x^T\right)\bigg|_{\tau=-T} \tag{25}$$

Note that φ_{xx} does not have to be nonsingular. Furthermore, Eq. (12) implies that matrices $S^*(\tau, t)$ and $c^*(\tau, t)$ have to be governed by the following differential equations:

$$\frac{\partial S^*(\tau, t)}{\partial \tau} = -A^T S^* - S^* A - C \tag{26}$$

$$\frac{\partial c^*(\tau, t)}{\partial \tau} = -A^T c^* \tag{27}$$

The estimate of the state is updated by

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left(S^* - \eta_{xx}\right)^{-1} \times \left[S^* f + \left(\eta_{xx}\frac{\mathrm{d}y}{\mathrm{d}t} + \eta_{xu}\frac{\mathrm{d}u}{\mathrm{d}t} + H_x^T - c^*\right)\right]\bigg|_{\tau=0} \tag{28}$$

In the case with $\eta = 0$, Eq. (28) is simplified as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f[x(t), u(t)] + S^{*-1}(0, t)\left\{L_x^T[x(t), y(t), u(t)] - r^*(0, t)\right\} \tag{29}$$

The matrix $S^* - \eta_{xx}$ has to be nonsingular at $\tau = 0$ in the present algorithm, though φ_{xx} can be singular. For a fixed time t, the second variation of the performance index is expressed as

$$\delta^2 \bar{J} = \tfrac{1}{2}\delta x^T\left(\eta_{xx} - S^*\right)\delta x\big|_{\tau=0} \tag{30}$$

where δx denotes arbitrary infinitesimal variation in the estimate of the state. Therefore, the positive definiteness of $\eta_{xx} - S^*$ is sufficient for local minimum of the solution of the TPBVP and also guarantees that the algorithm can be executed.

The matrix $S^*(0, t)$ is obtained by solving Eq. (26) with the boundary condition Eq. (24) as follows:

$$S^*(0, t) = -\Phi^T(-T, 0, t)\varphi_{xx}\big|_{\tau=-T}\Phi(-T, 0, t) - \int_{-T}^{0} \Phi^T(\tau, 0, t) C(\tau) \Phi(\tau, 0, t)\,\mathrm{d}\tau \tag{31}$$

where Φ denotes the transition matrix governed by

$$\frac{\partial}{\partial \tau}\Phi(\tau, \tau_0, t) = A(\tau, t)\Phi(\tau, \tau_0, t); \qquad \Phi(\tau_0, \tau_0, t) = I \tag{32}$$

When $C = H_{xx}$ is positive semidefinite, the nonsingularity of the second term on the right-hand side of Eq. (31) is equivalent to the nonsingularity of the observability Gramian of $(C^{1/2}, A)$ over $[-T, 0]$. Therefore, the nonsingularity of the observability Gramian of $(C^{1/2}, A)$ is sufficient for nonsingularity of $S^*(0, t)$, but not necessary because of the existence of the first term in the right-hand side of Eq. (31). Specifically, if the observability Gramian of $(C^{1/2}, A)$ is positive definite, then $\eta_{xx} - S^*$ is positive definite at $\tau = 0$, i.e., the present algorithm is executable and gives a local minimum solution.

Example

Experimental Model

The proposed method is applied to a simplified experimental model (Fig. 1) of a space vehicle to verify that it can be implemented for real-time state estimation. The experimental model represents a simplified space vehicle that is equipped with two thrusters and moves in plane. Since the experimental model can not move in the lateral direction, it is essentially a kind of mobile robot. The model is driven by two dc motors in the horizontal plane. The motors are inclined so that the edges of their shafts contact with the floor to generate thrust. The position and attitude of the experimental model are measured at the sampling interval of 0.05 s by a position sensor that tracks positions of two light-emitting diodes (LEDs) on the experimental model. The positions of the LEDs are transferred to a digital computer through an RS-232C interface. The digital computer (CPU: Cx486DRx2, NDP: Cx87DLC, clock 33 MHz) processes the sensor data and outputs signals to the model through a D/A converter. The output voltages from the D/A converter are put into a power booster that supplies current to the dc motors. The dc motors are modeled to rotate at velocities that are proportional to the input voltage. The actual velocities of the motors track the command velocities with time delays of about 0.2 s, which justifies the simple modeling of the motors in the present experiment.

Under the assumption that the velocities of the two thrusters are given as the input, the state variables of the space vehicle model are its position (x_1, x_2) and its attitude angle x_3 (see Fig. 2). The mathematical model of the model is expressed in the following affine system:

$$\frac{dx}{dt} = G(x)u \tag{33}$$

$$y = Cx \tag{34}$$

where $x = [x_1 \;\; x_2 \;\; x_3]^T$ denotes the state vector, $u = [u_1 \;\; u_2]^T$ the input vector, $y = [y_1 \;\; y_2]^T$ the measured output, and

$$G(x) = \frac{1}{2}\begin{bmatrix} -\sin x_3 & -\sin x_3 \\ \cos x_3 & \cos x_3 \\ 1/L & -1/L \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{35}$$

The measured output of the system is supposed to be the position of the model in the estimation problem, though the attitude angle can also be measured. The attitude angle is estimated by the proposed receding-horizon state-estimation algorithm and is compared with the measured attitude angle. If the two thrusters generate the same positive velocity as each other, the direction of the movement is identical to the attitude angle. Therefore, it is apparent that the attitude angle can be estimated from the input and measured output in the past.

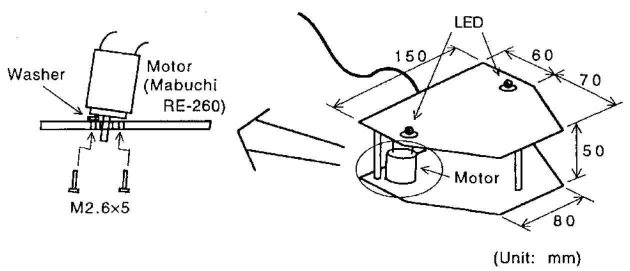

Fig. 1 Experimental model.

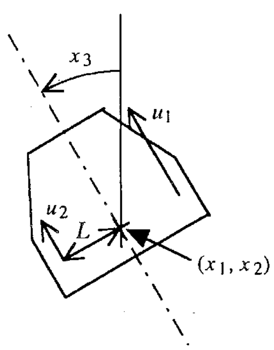

Fig. 2 Space vehicle model.

We identify the state space with the Euclidean space $\boldsymbol{R}^3$ to avoid the necessity to switch the local coordinates. Therefore the attitude angle x_3 is not restricted to the range of $0 \leq x_3 < 2\pi$ and can be any real number. It is straightforward to show that the observability codistribution[10] $\mathrm{d}O(x)$ the model is given by

$$\mathrm{d}O(x) = \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \sin x_3 \mathrm{d}x_3, \cos x_3 \mathrm{d}x_3\} \tag{36}$$

and $\dim \mathrm{d}O(x) = 3$ for any $x \in \boldsymbol{R}^3$. The fundamental result of nonlinear control theory[10] implies that the present model is locally observable on the whole state space, which agrees with physical intuition.

The performance index of the receding-horizon state-estimation problem is chosen as the following quadratic performance index:

$$J = \frac{1}{2}\int_{t-T}^{t} [y(\tau) - Cx(\tau)]^T Q [y(\tau) - Cx(\tau)] \,\mathrm{d}\tau \tag{37}$$

where $Q = \mathrm{diag}(q_1, q_2) > 0$ is a weighting matrix, and the horizon T is given by

$$T(t) = \begin{cases} t & (T_0 \leq t \leq T_f) \\ T_f & (t > T_f) \end{cases} \tag{38}$$

The receding-horizon state estimation starts with a nonzero initial horizon T_0, in contrast to the receding-horizon control, which is started with zero horizon.[1,2] The nonzero initial horizon is required in the receding-horizon state estimation, because the matrix $S^*(0, t)$ is singular and the algorithm cannot be executed when the horizon is zero. The algorithm is simplified during $T_0 \leq t \leq T_f$, because $\mathrm{d}T/\mathrm{d}t = 1$ and some terms vanish in Eqs. (18) and (25). The state is estimated by integrating Eq. (1) during $0 \leq t \leq T_0$.

The penalty φ is chosen as zero and φ_{xx} is singular in the present problem. Therefore, the algorithm in Eq. (29) is employed.

Simulation Results

The initial state is given in the simulation as $x_1 = -0.3$ m, $x_2 = -0.2$ m, and $x_3 = 0$ rad. The weighting matrix is set as $Q = \mathrm{diag}(q_1, q_2) = \mathrm{diag}(1, 1)$ through the simulation. Residuals in y_1 and y_2 are weighted equally ($q_1 = q_2$) because they are measured in the same accuracy in the experiment, and q_1 and q_2 are normalized to one because only the ratio of q_1 to q_2 affects the performance of the state estimation, as is apparent from Eq. (37). The parameter is chosen as $\zeta = 10$ for the stabilized continuation method. The input to the system is given by

$$u(t) = \begin{bmatrix} \alpha \sin \omega t \\ \alpha \sin(\omega t + \beta) \end{bmatrix} \tag{39}$$

where parameters are chosen as: $\alpha = 0.2$ m/s, $\omega = 0.2$ rad/s, and $\beta = 1$ rad, respectively. Some different situations are compared so as to examine the sensitivity of the proposed state estimation algorithm to the initial error in the estimated state and measurement noise.

The matrix $S^*(0, t)$ in the present case has the following specific structure:

$$S^*(0, t) = \begin{bmatrix} -q_1 T & 0 & * \\ 0 & -q_2 T & * \\ * & * & * \end{bmatrix} \tag{40}$$

where $*$ is an unspecified entry. The interlacing eigenvalues theorem for bordered matrices[11] implies

$$\lambda_1 \geq \max\{-q_1 T, -q_2 T\} \geq \lambda_2 \geq \min\{-q_1 T, -q_2 T\} \geq \lambda_3 \tag{41}$$

where λ_i $(i = 1, 2, 3)$ are the eigenvalues of $S^*(0, t)$ arranged in decreasing order. Therefore, at least two eigenvalues of $S^*(0, t)$ (λ_2

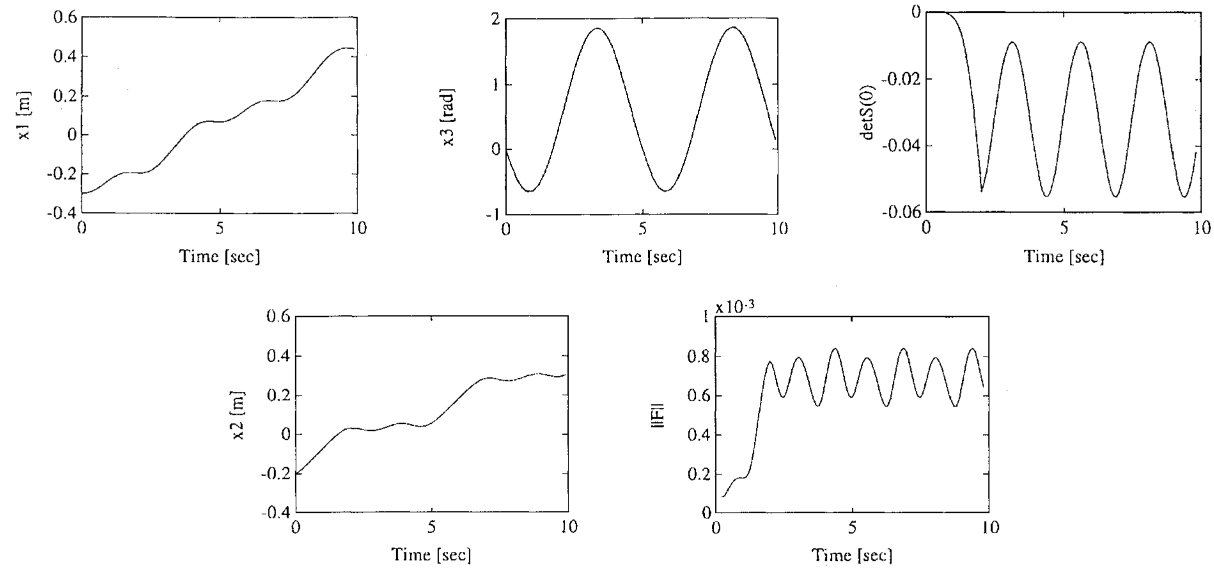

Fig. 3 Simulation result with $T_0 = 0.2$ s: estimated state variables (solid lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$. The estimated state variables coincide with the actual state variables.

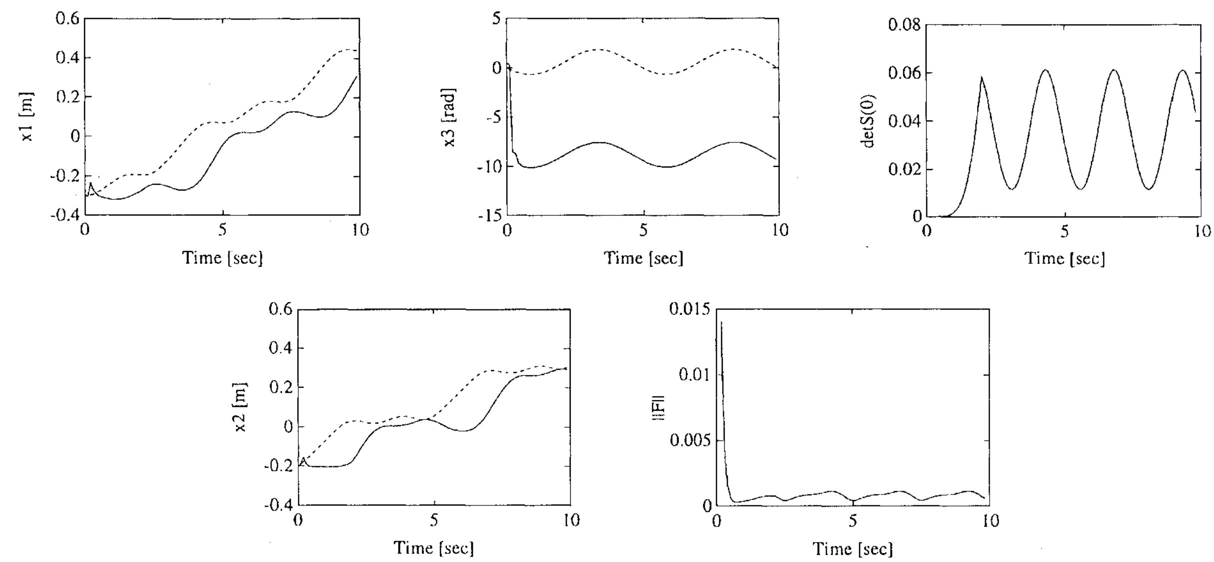

Fig. 4 Simulation result with $T_0 = 0.2$ s and error of 0.5 rad in the initial attitude: estimated state variables (solid lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

and λ_3) are negative, and $\text{sign}(\lambda_1) = \text{sign}[\det S^*(0, t)]$ holds. That is to say, $S^*(0, t)$ is negative definite if and only if $\det S^*(0, t)$ is negative. The discussion in the preceding section implies that a solution of the TPBVP gives a local minimum solution if $S^*(0, t)$ is negative definite. If $\det S^*(0, t)$ is positive, $S^*(0, t)$ is indefinite with two negative eigenvalues and one positive eigenvalue.

A simulation result is shown in Fig. 3, where the initial state is assumed to be known exactly. The initial horizon T_0 and the final horizon T_f are chosen as $T_0 = 0.2$ s, and $T_f = 2$ s, respectively. The estimated state (solid line) and the actual state (broken line) coincide. The error in the optimality condition $\|F\|$ is less than 0.001, and $\det S^*(0, t)$ is negative throughout the simulation. The proposed optimal state estimation algorithm works successfully giving a local minimum solution in this case.

In the second case, an initial error in the estimation is imposed to examine the sensitivity of the proposed algorithm. The simulation result in Fig. 4 shows a case with initial error of 0.5 rad in the attitude angle. The proposed algorithm fails to follow the actual state because of the initial error. Although the error in the optimality condition attenuates exponentially, which shows the algorithm is executed correctly, the estimated state gives just a stationary solution not a local minimum solution, since $\det S^*(0, t)$ is positive. The present algorithm allows only a small initial error. This sensitivity may be attributed to the fact that $S^*(0, t)$ is nearly singular at the beginning of the estimation when the horizon T is small.

As the third case, random measurement noises are included in the simulation. Random values with amplitude of 0.05 are added on the output variables y_1 and y_2. The simulation result in Fig. 5 shows that the proposed algorithm also fails in this case. It is observed from Fig. 5 that the estimation algorithm tries to follow even noisy measurements at the beginning of the estimation when the horizon T is small.

As the fourth case, the initial horizon T_0 is increased to 2 s, which is identical to the final horizon T_f. The simulation result in Fig. 6 shows that the estimation algorithm works well in spite of the initial error of 0.5 rad and the measurement noises. The error in the optimality condition $\|F\|$ decreases immediately, and the measurement noises are smoothed by the estimation algorithm. This smoothing property of the algorithm reduces the sensitivity to the measurement noises. It is also observed that $|\det S^*(0, t)|$ is large enough at the beginning of the estimation in Fig. 6 compared to preceding cases. Therefore, the estimation algorithm does not fall into numerical difficulty at the beginning of the estimation and is not sensitive to the initial error.

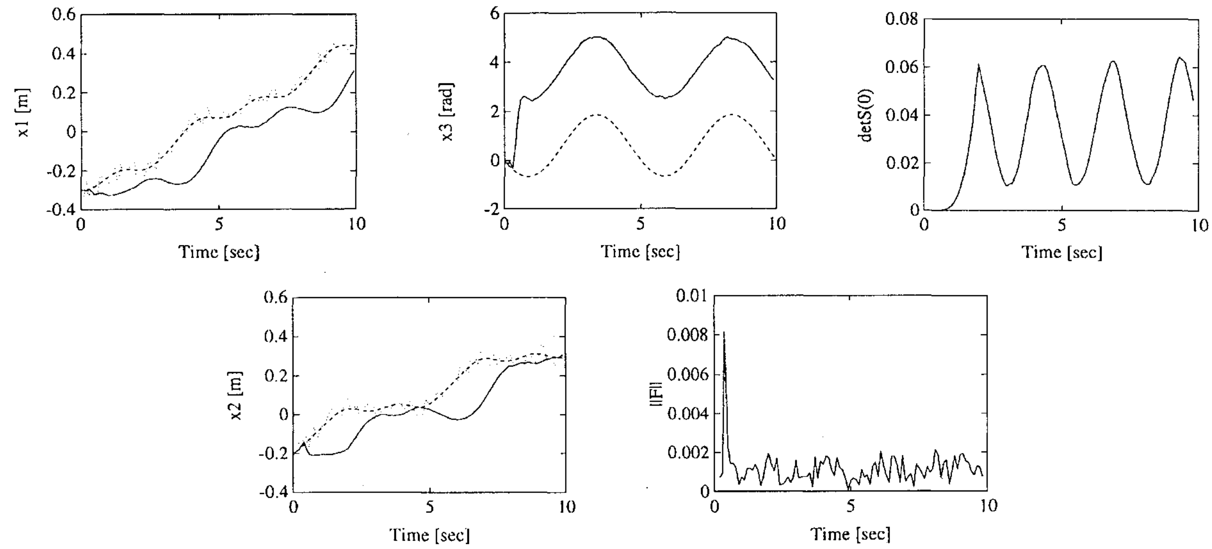

Fig. 5 Simulation result with $T_0 = 0.2$ s and random measurement noises: estimated state variables (solid lines), measured outputs (dotted lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

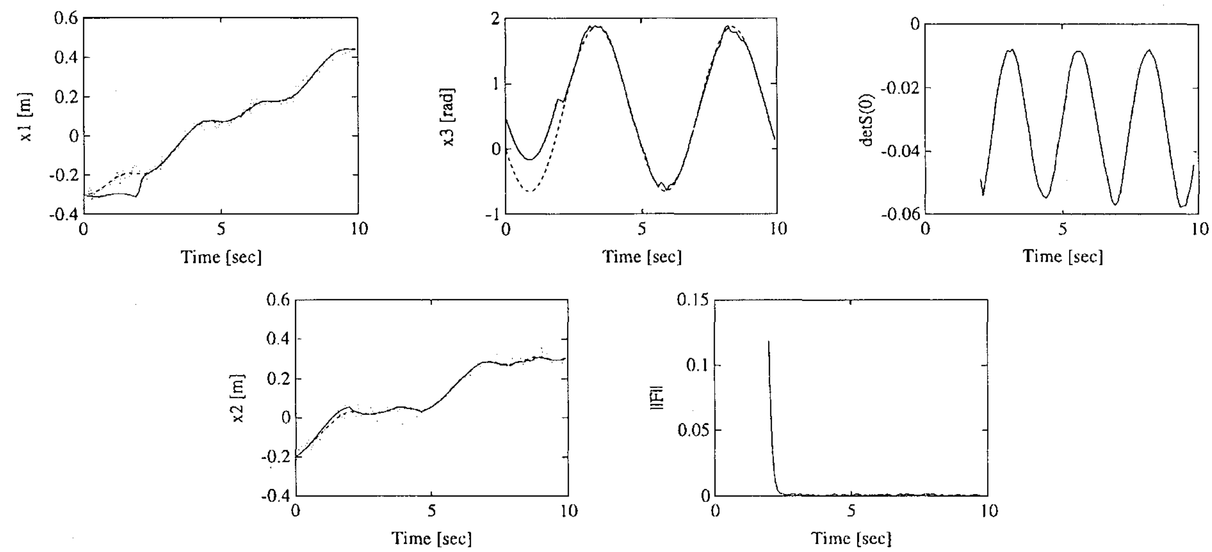

Fig. 6 Simulation result with $T_f = T_0 = 2$ s, error of 0.5 rad in the initial attitude, and random measurement noises: estimated state variables (solid lines), measured outputs (dotted lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

It is concluded based on these simulation results that the proposed optimal state estimation algorithm is sensitive to the initial error in estimation and the measurement noises when the horizon is small. The initial horizon T_0 should be chosen as large as possible to reduce the sensitivity of the estimation algorithm to the initial error and the measurement noises.

Robustness to Modeling Error

Next, the robustness property of the algorithm is tested numerically with respect to error in the input gain. The actual input to the experiment model is given by $k_u \boldsymbol{u}$ where k_u denotes a scalar coefficient and $\boldsymbol{u}$ is the input vector used in the estimation algorithm. The initial horizon T_0 and the final horizon T_f are set as $T_0 = T_f = 2$ s. The initial estimation error of 0.5 rad in the attitude angle and the measurement noises are also imposed in the simulation.

Figure 7 shows a simulation result of a case with $k_u = 0.9$. The actual magnitude of the input is smaller than the magnitude in the estimation algorithm in this case. In comparison with Fig. 6, it is shown in Fig. 7 that the algorithm fails because of the error in the magnitude of the input. In contrast to Fig. 7, the algorithm does not fail in Fig. 8, which shows a case with $k_u = 2$. It may be concluded from the simulation results that the present estimation algorithm is robust against an increase in magnitude of the input, though it is sensitive to a decrease in magnitude of the input. The algorithm fails if the state varies more slowly than predicted based on a mathematical model. However, the estimation algorithm can follow the state that varies faster than predicted.

Experimental Result

The proposed state estimation algorithm is implemented on the digital computer for the hardware experiment by discretizing the differential equations with the forward difference of the 0.05-s sampling interval. To avoid failure of the estimation algorithm caused by the modeling error in the experiment, the estimation algorithm is modified slightly based on the observation in the preceding simulation. That is to say, in the estimation algorithm, the magnitude of input is decreased to be 0.625 (=1/1.6) of the actual magnitude ($k_u = 1.6$). The actual amplitude of the input is 0.2 m/s, which is the same as the amplitude depicted in Figs. 3–6. Figure 9 shows an experimental result with the initial estimation error of 0.5 rad in the attitude angle. The time history of the actual state in Fig. 9 does not agree well with the simulation results, which shows the existence of modeling error in the experiment. It is shown in Fig. 9 that the estimated state follows the actual state in spite of the initial estimation error and the modeling error. The error in the optimality condition decreases immediately below 0.01, which shows that the implemented algorithm gives a solution of satisfactory accuracy. It may be concluded from the experimental result that the proposed

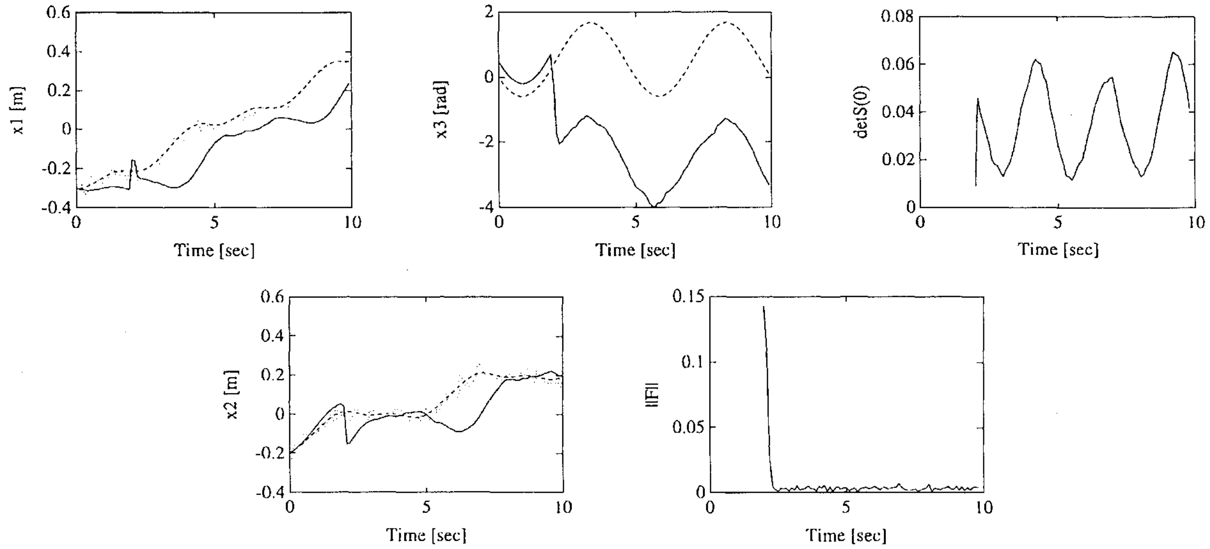

Fig. 7 Simulation result with $T_f = T_0 = 2$ s, error of 0.5 rad in the initial attitude, random measurement noises, and smaller magnitude of actual inputs ($k_u = 0.9$): estimated state variables (solid lines), measured outputs (dotted lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

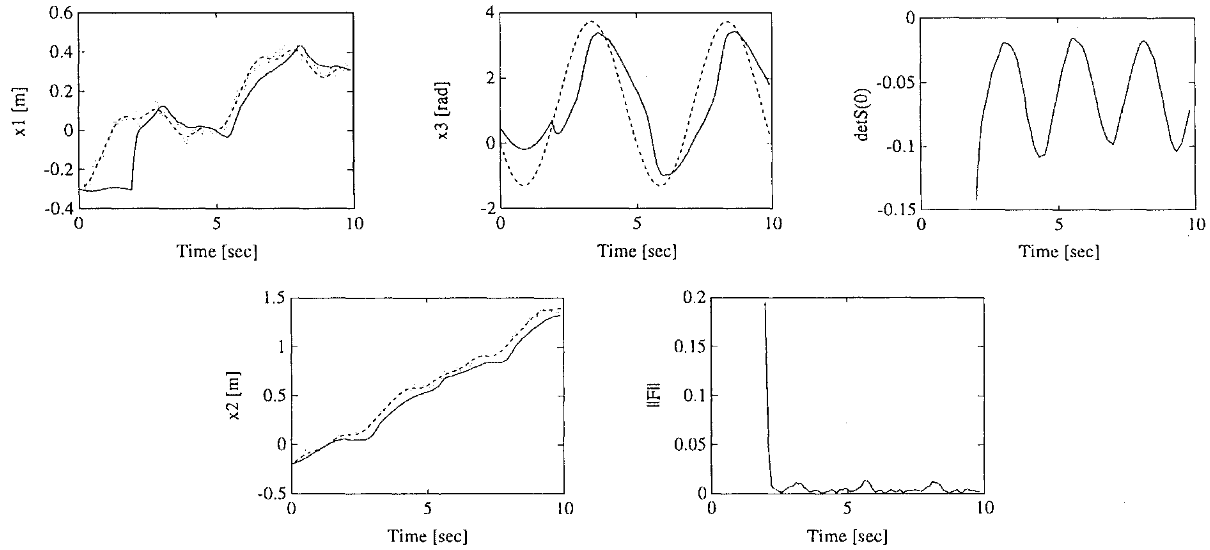

Fig. 8 Simulation result with $T_f = T_0 = 2$ s, error of 0.5 rad in the initial attitude, random measurement noises, and larger magnitude of actual inputs ($k_u = 2$): estimated state variables (solid lines), measured outputs (dotted lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

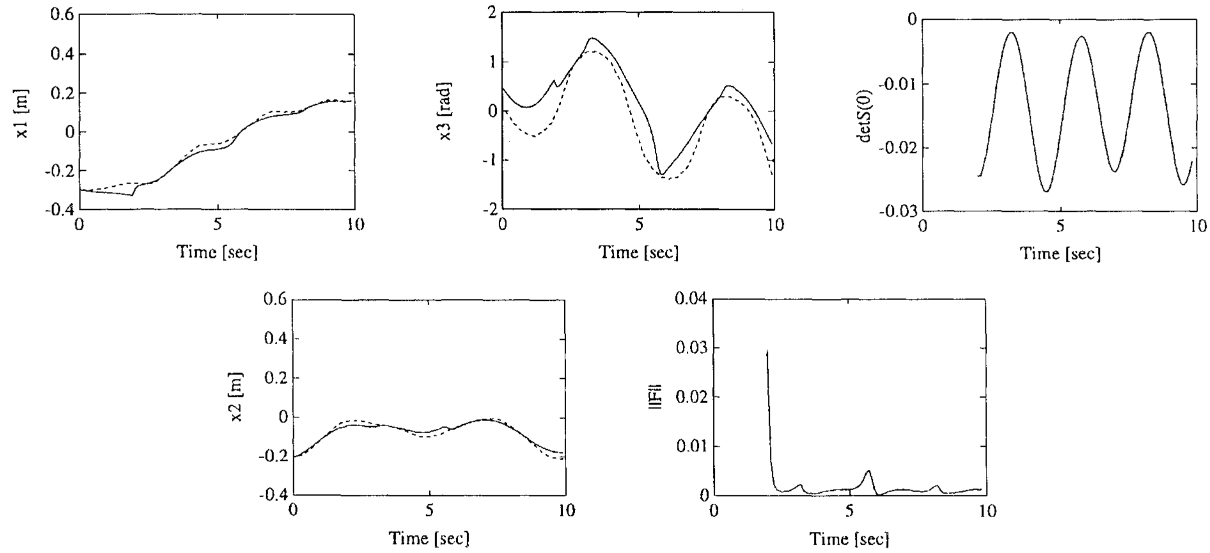

Fig. 9 Experimental result with $T_f = T_0 = 2$ s, error of 0.5 rad in the initial attitude, and larger magnitude of actual inputs ($k_u = 1.6$): estimated state variables (solid lines), actual state variables (broken lines), the error in the optimality condition, and the determinant of the matrix $S^*(0, t)$.

receding-horizon state-estimation algorithm can be implemented for real-time nonlinear state estimation.

Conclusions

A real-time optimization technique is proposed for optimal state estimation of general nonlinear systems. The receding-horizon performance index is employed to determine the optimal state estimation algorithm as the dual of the receding-horizon state-feedback control.

The stabilized continuation method is applied to solve the receding-horizon state-estimation problem in real time. The two-point boundary-value problem associated with the optimal estimation problem is converted to a nonlinear algebraic equation with respect to the present state, and its solution is traced with respect to time. The present solution technique does not involve any approximation or iterative algorithms in principle. The discretization of the algorithm is the only approximation required in the actual implementation on digital computers. Therefore, the present algorithm is suitable to real-time optimal state estimation. It is shown that the proposed algorithm requires only a weaker assumption than the observability of the system. It is also shown by analyzing the second variation of the performance index that the proposed algorithm is executable and gives a local minimum solution if a certain matrix is positive definite.

The proposed solution technique is applied to a simplified model of a space vehicle. Although the dynamics of the model are simplified, the model is a nonlinear nonholonomic system. As a result of the present numerical simulation, it is concluded that the proposed algorithm is sensitive to the initial estimation error and the measurement noises when the initial horizon is small, although the algorithm works well in an ideal situation without any initial error or noises. The initial horizon should be chosen as large as possible to reduce the sensitivity of the estimation algorithm to the initial error and the measurement noises. When the initial horizon is large enough, the algorithm works well in spite of the initial error and the measurement noises. The robustness property is also examined against error in the input gain of the model. Simulation results show that the present algorithm does not fail even if the actual magnitude of the input is larger than the mathematical model, though it fails if the actual magnitude is smaller.

Based on the observation obtained in the numerical simulation, a hardware experiment is performed with the proposed algorithm that is modified to avoid failure caused by modeling error. The algorithm is implemented on a digital computer and is shown to give satisfactory estimation in spite of simplifications of the model. It may be concluded that the proposed algorithm is suitable for real-time state estimation of nonlinear systems. In principle, nonlinear optimal output feedback control and nonlinear optimal adaptive control are possible by combining the algorithms of the nonlinear receding-horizon state feedback and state estimation. Integration of the nonlinear state feedback and nonlinear state estimation is recommended for future study.

Acknowledgment

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